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EXPERIMENTS ON THE KINEMATICS OF LARGE PLASTIC STRAIN IN ORDERED SOLIDS

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Abstract—On thin-walled tubes, for finite plastic strain the precise measurement—to four decimal places—of length, outside diameter, inside diameter, the uniform helices formed by initially straight generators on the surface of twisted cylinders, and the total angle of twist, becomes the basis for a study of the kinematics of very large plastic deformation. The data are obtained from over 200 thin-walled tubes of several ordered solids, including metal alloys, that have been twisted from small angles to angles as large as 360°, and extended by axial strains from small to 30% or more. The laboratory discovery is made that even at deformation of this magnitude, rigid body rotation of principal axes is minuscule, a negligible phenomenon. Adding to these data the measured applied moments, internal pressure, and axial forces, provides a direct calculation of Cauchy stress and corresponding natural or deviatoric strain in the deformed reference configuration. From these details, the rigid body rotation **R** of principal axes between deformed and undeformed reference configurations, the deformation gradient **F**, the homogeneous deformation **V**, the change in volume due to plasticity, the degree of isotropy or anisotropy, and the applicable internal constraint, can be determined directly for arbitrary loading paths without reference to any particular theory of finite strain plasticity.

1. INTRODUCTION

In the recent past, finite plastic deformation in ordered solids has been the subject of diverse conjecture and numerous theories. In an effort to delimit such conjecture and provide an independent foundation for proper theory, this paper describes the state of a grossly deformed solid by a measurement to four decimal places of *all* dimensions while at maximum deformation. A kinematical analysis of the observed deformed state is independent of the details of the deformation history that provided such a configuration.

Cylindrical thin-walled tubes for these studies have a length to mean diameter ratio of 10 with a wall thickness of from 10 to 15% of the mean radius, depending on the circumstances. The tubes are loaded by a measured arbitrary combination of axial force, torsional torque, and internal pressure. Measurements are made of the elongation in the axial direction while under load, the total angle of twist also while under load, and, of special concern for the kinematical matters of interest here, the pre-deformation and post-deformation outside and inside diameters, and the spacing of inscribed helices made by lines initially parallel to the axis on the surface of the undeformed cylinders.

The measurements of outside diameters are made in orthogonal directions at five locations along the tube. This provides, in each instance, a total of ten measurements to assure that the cylinders were originally circular and remained so at large deformation. In the central region the inside diameters are measured by point to point telescope gages inserted in four radial directions at each open end of the tube. (See Appendix and Table 1.)

For proportional loading at any ratio of internal pressure and axial force, since the principal axes do not rotate, the prediction and description of response offer no difficulties when first Piola–Kirchhoff stress components are compared with geometric strain components, both defined with respect to the original, undeformed reference configuration (Bell, 1988a).

Let **R** designate a rigid body rotation of principal axes between the deformed and undeformed reference configurations, where $\mathbf{R}^{-1} = \mathbf{R}^{T}$. Kinematical problems arise in determining **R** when the tubes are subjected to torsional twist, particularly when twisting occurs with simultaneous axial elongation and hence combines with the finite plastic strain deformation that is observed when rotations are absent.

The finite plastic strain is indeed large. The tubes are elongated by an increment in excess of one-third of their original length. They are twisted a complete turn of 360°. The bounds are experimental.

In analyzing these data, the purpose is to explore kinematical underpinnings, independent of any particular continuum theory of finite strain plasticity. When these kinematical details have been established, however, it is pertinent to pay heed to the logical restrictions they impose on the choice of continuum theory in particular, and on assumptions in general.

Since the time of my first incremental wave experiments in finite strain plasticity in the summer of 1949, I have maintained a library of deformed specimens that, together with the detailed written records of each test, provide a basis for the review of earlier data, such as that described below. Beginning in September 1958, all my tests, or those of my graduate students, have been and still are numbered consecutively in order that they may be referred to from one study to another or from one paper to another. From the nearly 300 thin-walled tubes for 11 metals and metal alloys that constitute approximately 20% of this specimen library for the interval between 1967 and the present, I have chosen as representative and of particular interest for the present discussion, 31 tubes the responses of which have been described in great detail in three recent papers.

All the copper tests included below except for tests 2316, 2317, and 2319, are described in Bell and Khan (1980). These three copper tests are examples from the cyclical loading series described in Bell (1983a). The 1020 annealed mild steel tests are described in Bell (1983b). By choosing these tests among the many that also have been similarly analyzed, one omits a repetition of detail here, and refers to those recent sources for the detailed descriptions of the finite strain response for any given test.

An unequivocal, complete description of the deformed state for large finite strain permits a kinematical analysis independent of stress. That the kinematical analysis is independent of stress infers, too, that it is independent of whatever theory of finite strain plasticity has been invoked to characterize observation. On the other hand, given the loads, with *all* specimen dimensions in the deformed state being measured, it becomes possible to determine the Cauchy stress and the corresponding strain in the deformed reference configuration.

2. THE KINEMATICS OF "SIMPLE" TORSION AT FINITE PLASTIC STRAIN

One begins with the tabulation (in Table 1) of data on tubes subjected only to a twisting torque. In comparing these data with those in Table 2 for which large axial forces

Table 1

Test	Solid	L_0 pre-deformation (in.)	L_{PD} post-deformation (in.)	Initial outside diameter (in.)	Outside diameter maximum deformation (in.)	Initial inside diameter (in.)	Inside diameter maximum deformation (in.)	Maximum twist, θ (deg.)	Maximum shear, s_{v_1}
2230	Fe	4.277	Buckled	0.4143	0.4145	0.3755	0.3750	115	0.093
2258	Fe	4.219	4.188	0.4401	0.4394	0.3750	0.3753	103	0.086
2259	Fe	4.219	4.172	0.4398	0.4403	0.3750	0.3756	195	0.167
2260	Fe	4.203	4.170	0.4396	0.4389	0.3750	0.3754	147	0.127
2261	Fe	4.219	4.156	0.4397	0.4394	0.3750	0.3747	280	0.238
2264	Fe	4.219	4.192	0.4399	0.4389	0.3750	0.3744	229	0.197
2265	Fe	4.219	4.182	0.4400	0.4411	0.3750	0.3750	283	0.243
2283	Fe	4.219	Buckled	0.4150	—	0.3760	0.3758	108	0.068
1832	Cu	4.125	4.078	0.4410	0.4409	0.3755	0.3753	148	0.128
1760	Cu	4.250	4.203	0.4234	0.4294	0.3750	0.3755	121	0.101
1763	Cu	4.187	4.141	0.4205	0.4201	0.3763	0.3747	165	0.137
1778	Cu	4.187	Buckled	0.4185	0.4152	0.3765	0.3753	207	0.169
Total average 11 tests				0.4334	0.4333	0.3752	0.3753		

Table 2. (See Appendix for more detail.)

Test	Maximum axial strain, E_{xx}	Total angle of twist, θ (deg.)	Circumferential strain o.d. E_r(max)	Circumferential strain i.d. E_r(max)	Maximum shear strain, s_{yx}	Solid	Load path
1812	0.043	199	−0.023	−0.021	0.182	Cu	P
2211	0.046	297	−0.023	−0.033	0.274	Cu	NP
1815	0.048	201	−0.022	−0.028	0.181	Cu	NP
2269	0.049	113	−0.024	−0.028	0.104	Fe	NP
1813	0.068	179	−0.030	−0.032	0.145	Cu	P
1805	0.075	150	−0.033	−0.037	0.138	Cu	NP
1799	0.076	163	−0.040	−0.045	0.149	Cu	NP
2270	0.092	302	−0.048	−0.052	0.283	Fe	NP
1806	0.093	154	−0.047	−0.049	0.141	Cu	NP
2316	0.106	†	−0.059	−0.060	†	Cu	NP
2332	0.109	9	−0.048	−0.055	0.007	Fe	P
2317	0.119	†	−0.060	−0.061	†	Cu	NP
2319	0.134	134	−0.065	−0.068	0.147	Cu	NP
2286	0.137	114	−0.068	−0.068	0.103	Fe	P
2167	0.180	144	−0.089	−0.093	0.129	Fe	P
2169	0.204	214	−0.104	−0.103	0.193	Fe	NP
2210	0.218	66	−0.102	−0.106	0.061	Cu	P
2271	0.240	347	−0.119	−0.122	0.292	Fe	NP
2262	0.246	176	−0.117	−0.126	0.204	Fe	P
1974	0.197	0	−0.102	−0.097	0.000‡	Cu	P

† Tests 2316 and 2317 were cyclically loaded many times in large clockwise and counterclockwise torsion during increasing axial strain. The absolute difference in measured torsional strain is 0.260 (Bell, 1983a).
‡ Pure tension.

accompanied the twist, one must remember that for the tests in Table 1 no axial loads were imposed. The following were measured: angle of twist *determined while under load*, the maximum shear strain in the undeformed reference configuration, s_{yx}, determined under load from $s_{yx} = [R_m\theta]/L_0$, the pre- and post-deformation inside and outside diameters, and tube lengths. The quantities R_m, L_0, and θ are the mean undeformed radius, the undeformed length, and the total angle of twist. The pre-deformation inside diameters were precision reamed with a special tool to 0.3750 in. with the dimension confirmed by a similarly precise internal, 0.3730–0.3770 in. dial gage, inserted into the central regions of the tube.

As the data in Table 1 indicate, the outside and inside diameters remain constant. There is no change in the mean diameter or cross-sectional area of the tube, irrespective of the magnitude of the angle of twist. Moreover, the helices formed from initially inscribing parallel generators on the surface of the undeformed cylinders remain smooth and evenly spaced.

In Table 1 the averaged deformed outside diameter for an average angle of twist of 174 differs by 0.0001 in. from the averaged pre-deformation diameter. For inside diameters, the difference in the averages is also 0.0001 in. Seventeen of the 23 diameter comparisons have individual differences of well under 0.0010 in. with an average individual difference of less than 0.0004 in. The average individual difference for all 23 comparisons is 0.0009 in.

To emphasize that the maximum angles of twist are close to failure, one notes that the specimens of tests 2230, 2283, and 1778, with a wall thickness of only 10% of the mean radius, all buckled. Test 2265 was in the process of buckling as the tube was unloaded. Specimens with a wall thickness of 15% of the mean radius underwent gross buckling when twisted just above the 283° maximum of test 2265.

Of equal kinematical significance is the presence of a reverse version of the Wertheim–Poynting effect. The length of the specimen, L, decreases a small amount, $E_{xx} = \Delta L/L_0$. Such a decrease in length during simple twisting makes evident that for thin-walled tubes the finite deformation of simple twisting is *not* in simple shear. Since the cross-sectional area is unchanged even for a total twist angle of $\theta = 283°$, such a decrease in length is incompatible with the commonly assumed internal constraint of isochoric deformation or incompressibility in finite strain plasticity.

3. THE DEFORMED STATE FOR COMBINED TENSION–TORSION

In Table 2 are 19 tests on annealed copper and annealed mild steel, listed in the order of the maximum measured axial strain, from 4.3 to 24.6%. The first column gives the test number that may be used to locate the illustrative tests, first cited in Bell and Khan (1980) and Bell (1983a, b). The second column is the maximum axial strain, E_{xx}, measured while under load. The third shows the total angle of twist, θ, also measured under load. The next two columns tabulate the maximum circumferential strain, $E_r = (D_0 - D)/D_0$, determined, respectively, from comparing outside diameters before and after deformation and from comparing inside diameters before and after deformation. (See Appendix for the detailed listing of the data for inside and outside diameters.) The sixth column tabulates the maximum shear strain in the undeformed reference configuration, $s_{yx} = R_m\theta/L_0$. The remaining two columns indicate, respectively, the type of solid and whether the loading path is proportional or nonproportional. As will be discussed in detail below, due to unloading, the circumferential strain determined from the inside diameter differs slightly from that determined from the outside diameter. The average is used in the analysis below. As will be shown, the final results are relatively insensitive to this choice of measurement.

These data provide a complete description of the shape of the deformed tube, including the change that has occurred in wall thickness. Test 2271 is for a mild steel tube twisted nearly a complete turn, $\theta = 347°$, and at the same time axially extended by 24%. This specimen was loaded first in tension alone then in torsion at constant tension, a non-proportional stress path. In the tests in Table 2, the variety of maxima in both extension and shear encompasses the range of finite deformation. At or beyond the largest values, the specimens either neck in tension or buckle in torsion. The largest angle of twist obtainable in any solid studied has been $\theta = 360°$ accompanied by 30% axial strain. However, as will be shown below, the largest rigid body rotation of principal axes in Fe and Cu occurs not for test 2271 at $\theta = 347°$ twist but for test 2211 in which the axial strain is only 4.6% and the angle of twist reaches only $\theta = 297°$.

4. A KINEMATICAL ANALYSIS OF LARGE FINITE PLASTIC DEFORMATION†

Introduce the polar coordinates R, Θ, and Z on original cylinder. Let

$$\alpha = D/D_0 = 1 + E_r$$

$$\gamma = R_m\theta/L_0 = s_{yx}$$

$$\delta = 1 + \Delta L/L_0 = 1 + E_{xx}$$

where D, ΔL and θ are all determined from measurements in Tables 1 and 2, and R_m, L_0, and D_0 are the pre-deformation mean radius, length, and diameter. For the deformation, let

$$r = \alpha R \tag{1}$$

$$\theta = \Theta + (\gamma/R_m)Z \tag{2}$$

$$z = \delta Z. \tag{3}$$

Map to the plane $R/R_m \simeq 1$, then

$$x_1 = z = \delta X_1 \tag{4}$$

$$x_2 = r\theta = \alpha R[\Theta + (\gamma/R_m)Z] = \alpha X_2 + \alpha\gamma X_1 \tag{5}$$

† Once again I am indebted to J. L. Ericksen (Ericksen, 1987), in this instance for his suggestions on how best to interpret these data.

$$x_3 = r = \alpha X_3 \tag{6}$$

gives

$$F = \begin{pmatrix} \delta & 0 & 0 \\ \alpha\gamma & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} = VR. \tag{7}$$

For the two-dimensional problem, try

$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then the two-dimensional part of $\mathbf{FR}^{\mathrm{T}} = \mathbf{V}$ must be symmetric

$$V = \begin{pmatrix} \delta\cos\phi & -\delta\sin\phi & 0 \\ \alpha\sin\phi + \alpha\gamma\cos\phi & -\alpha\gamma\sin\phi + \alpha\cos\phi & 0 \\ 0 & 0 & \alpha \end{pmatrix}. \tag{8}$$

One needs $\alpha\sin\phi + \alpha\gamma\cos\phi = -\delta\sin\phi$, or

$$\tan\phi = -(\alpha\gamma)/(\alpha+\delta). \tag{9}$$

Also, from eqn (8) one may write

$$\begin{aligned} \text{trace } \mathbf{V} &= \alpha + \alpha\cos\phi - \alpha\gamma\sin\phi + \delta\cos\phi \\ &= \alpha + (\alpha+\delta)\cos\phi - \alpha\gamma\sin\phi \\ &= \alpha + (\alpha+\delta)[\cos\phi + \tan\phi\sin\phi] \end{aligned}$$

or

$$\text{trace } \mathbf{V} = \alpha + (\alpha+\delta)\sec\phi. \tag{10}$$

Introduce the geometric strain $\mathbf{E}$ where $\mathbf{E} = \mathbf{V} - \mathbf{I}$. Then

$$\text{trace } \mathbf{E} = \text{trace } \mathbf{V} - 3 = \alpha + (\alpha+\delta)\sec\phi - 3. \tag{11}$$

Finally, from eqn (8) one obtains III_V, from which any change of volume, ΔU, can be determined. In eqn (12), U_0, is the pre-deformation volume

$$\Delta U/U_0 = III_V - 1 = \alpha^2\delta - 1. \tag{12}$$

For α, γ, and δ provided by the measurements in Tables 1 and 2, one sees tabulated in Table 3 the angle of the rigid body rotation of principal axes, ϕ, from eqn (9), and trace **V** from eqn (10).

The most striking feature of these results is that although the average maximum angle of twist is half of a complete turn, $\theta = 176°$, the average angle, ϕ, for rigid body rotation of principal axes at that maximum is minuscule; it is only $\phi = -4.19°$. Even for an angle of twist of nearly a complete turn, simultaneously accompanied by an extension of 24%, the rigid body rotation is only $\phi = -6.91°$ as is seen in test 2271. The largest angle ϕ

Table 3

Test	Maximum twist, θ (deg.)	Maximum elongation, E_{zz} (%)	α	γ	δ	ϕ (deg.)	Trace V
2230	115	—	1.000	0.093	—	−2.66	—
2258	103	−0.7	1.000	0.086	0.993	−2.47	2.9950
2259	195	−1.1	1.001	0.167	0.989	−4.80	2.9960
2260	147	−0.8	1.000	0.127	0.992	−3.65	2.9960
2261	280	−1.5	0.999	0.238	0.985	−6.84	2.9992
2264	229	−0.9	0.998	0.197	0.991	−5.65	3.0007
2265	283	−0.9	1.001	0.243	0.991	−6.96	3.0058
2283	108	—	1.002	0.068	—	−1.95	—
1832	148	−1.1	1.000	0.128	0.989	−3.68	2.9931
1760	121	−1.1	1.008	0.101	0.989	−2.91	2.9916
1763	165	−1.1	0.999	0.137	0.989	−3.94	2.9937
1778	207	—	0.997	0.169	—	−4.83	—
1812	199	4.3	0.978	0.182	1.043	−5.03	3.0068
2211	297	4.6	0.972	0.274	1.046	−7.52	3.0075
1815	201	4.8	0.975	0.181	1.048	−4.99	3.0022
2269	113	4.9	0.974	0.104	1.049	−2.87	2.9995
1813	179	6.8	0.969	0.145	1.068	−3.95	2.9968
1805	150	7.5	0.965	0.138	1.075	−3.74	2.9968
1799	163	7.6	0.958	0.149	1.076	−4.01	2.9960
2270	302	9.2	0.951	0.283	1.092	−7.50	3.0096
1806	154	9.3	0.952	0.141	1.093	−3.76	3.0014
2332	9	10.9	0.949	0.007	1.109	−0.19	3.0006
2319	134	13.4	0.934	0.147	1.134	−3.80	3.0010
2286	114	13.7	0.932	0.103	1.137	−2.66	3.0032
2167	144	18.0	0.909	0.129	1.180	−3.21	3.0013
2169	214	20.4	0.897	0.193	1.204	−4.71	3.0041
2210	66	21.8	0.896	0.061	1.218	−1.48	3.0107
2271	347	24.0	0.879	0.292	1.240	−6.91	3.0094
2262	220	24.6	0.878	0.204	1.246	−4.84	3.0095
avg.	176				avg.	−4.19	3.0010

observed in any of the tests is that for test 2211. The rotation of principal axes is $\phi = -7.52°$ for a test in which the maximum angle of twist is smaller than that of test 2271, namely, $\theta = 297°$, with a maximum axial strain of just 4.6%. A similar angle is obtained for test 2270 at $\theta = 302°$ twist and only 9.2% extension. For annealed Cu and Fe, efforts to exceed the maxima in tests 2271, 2211, and 2270 resulted in either necking or buckling of the tube.

To emphasize the negligible role of the rigid body rotation in large finite deformation, one notes that for the average angle of $\phi = -4.19°$, one has $\cos\phi = 0.9973$, $\sec\phi = 1.0027$, and $\sin\phi = \tan\phi = -0.073 \simeq \phi$. For test 2169, where both the axial strain and shear strain are approximately 20%, the rotation angle is only $\phi = -4.71°$. All combinations of equal strain below 20% (such as that of test 2319 at a combination of strains of approximately 14% where $\phi = -3.80°$) emphasize the negligible role of rigid body rotation of principal axes, including finite strain at the obtainable maxima.

The first group of tests, from test 2230 through test 1778 in Table 3, are for twisting alone, in "simple" twisting. In each instance there is a small decrease in specimen length, as noted above. For a larger wall thickness, test 2265, the largest angle of twist is $\theta = 283°$; the rigid body rotation is only $\phi = -6.96°$. This was one of several tubes in which initially straight lines along the generators of the undeformed cylinder produced parallel, undistorted helices at maximum deformation.

A second striking feature of the results of the kinematical analysis tabulated in Table 3 is that the average value of trace V is 3.0010. Since trace E = trace V − 3 one has trace E = 0.0010 in eqn (11). This is in accord with results from the direct measurement of changes of volume given by the internal constraint trace E = 0 for stress paths in which there is no rigid body rotation. In the absence of rotation, the internal constraint, trace E = 0, has been observed from the *measured* diameters of thin-walled tubes in pure tension, solid cylinders and cubes in pure compression, and from displacements of cubes under load in the Bridgman two-dimensional compression experiment. There is no rotation of principal

Table 4†

Test	$\Delta U/U_0 = III_V - 1$ A	B	E_{xx} (%)	Test	$\Delta U/U_0 = III_V - 1$ A	B	E_{xx} (%)
1812	−0.002	−0.001	4.3	2332	−0.001	−0.009	10.9
2211	−0.012	−0.002	4.6	2319	−0.012	−0.013	11.9
1815	−0.004	−0.002	4.8	2286	−0.012	−0.013	13.4
2269	−0.005	−0.002	4.9	2167	−0.025	−0.013	13.7
1813	0.003	−0.003	6.8	2169	−0.031	−0.023	18.0
1805	0.001	−0.004	7.5	1974‡	−0.029	−0.027	19.7
1799	−0.012	−0.004	7.6	2210	−0.022	−0.029	20.4
2270	−0.012	0.006	9.2	2271	−0.041	−0.039	24.0
1806	−0.009	−0.008	9.3	2262	−0.039	−0.042	24.6

† Bauschinger (1879) was the first of many to measure changes of volume *during loading* for finite plastic strain in metals and other solids. As a result, he also was the first to find that unloading restored the original pre-deformation volume. A comparison of pre-deformation and post-deformation volumes was, and is, a poor indicator of incompressibility during plastic flow. One must measure volume *during* loading, not after unloading.

‡ See test 1974 in Table 2 for pure tension, where for trace $\mathbf{E} = 0$, one has $E_r = -E_{xx}/2$, or for $E_{xx} = 0.197$ measured, one has $E_r = -0.0985$ predicted and from Table 2 one has the post-deformation measurement 0.0995. Calculated for simple tension, for test 1974 the change of volume at maximum strain is $\Delta U/U_0 = -0.029$ as shown.

axes in the Bridgman pure shear compression experiment or in pure tension or compression (Bell, 1988a).

Given $\mathbf{V} = \mathbf{E} - \mathbf{I}$ one can see from eqn (14) that trace $\mathbf{E} = 0$ implies a change of volume during loading in the finite plastic strain domain

$$\Delta U/U_0 = \text{trace } \mathbf{E} - II_E + III_E = III_V - 1 \tag{13}$$

or

$$\Delta U/U_0 = -II_E + III_E. \tag{14}$$

From III_V, one may determine the Cauchy stress $\bar{\sigma}$ in terms of the internal constraint, trace $\mathbf{E} = 0$; $\bar{\sigma} = (III_V)^{-1}$, $\mathbf{F}\mathbf{T}_R^{\mathrm{T}} = (\alpha^2\delta)^{-1}\mathbf{V}\sigma$, where $\mathbf{T}_R^{\mathrm{T}}$ is the transpose of the first Piola-Kirchhoff stress tensor.

Under heading A in Table 4 are tabulated the changes of volume obtained from eqn (12) using the data of Table 3. For comparison, tabulated under heading B in Table 4 are the predicted changes of volume for simple tension determined from the internal constraint trace $\mathbf{E} = 0$ using eqn (14).

Beatty and Stalnaker (1986) have discussed the general implications of the fact that the present author's internal constraint, trace $\mathbf{E} = 0$, leads to a Poisson function of 1/2 in simple tension. In the present instance, in such terms, one has for the radial strain, $E_r = -E_{xx}/2$. Hence, eqn (14) becomes $\Delta U/U_0 = -(3/4)E_{xx}^2 + (1/4)E_{xx}^3$.

Both $\gamma = s_{yx}$ and $\delta = 1 + E_{xx}$ in Table 3 are determined while the tube is under load. As was indicated above, the quantity $\alpha = D/D_0$, is at present accurately determinable only after the load has been removed. During unloading there is a return to the pre-deformation volume; hence α is slightly modified. Direct experimental measurement during loading and unloading has shown that not only is the original volume recovered but also the details of recovery are not dimensionally uniform. The interesting general properties of this recovery of volume are the subject of current laboratory study, i.e. a study of constitutive equations for loading in the opposite direction.

Contrary to the classical expectation for unloading when one assumes the internal constraint of incompressibility, the measured unloading from finite plastic strain governed by the internal constraint trace $\mathbf{E} = 0$, is a small strain plastic recovery, not a linear elastic recovery. To observe and measure this, one must accurately compare specimen dimensions measured under load and during unloading, with dimensions measured when that load is completely removed. Bauschinger (1879) did this for tensile as well as for compression measurements on a cubical section of long rectangular bars of steel and other solids, using an optical technique which he developed, and in 1973 I introduced similar tests for the 25% axial compression of circular solid bars of copper (Section 4.35 of Bell (1973)). In a current paper (Bell, 1988a), the detail of this unloading behavior in cubical specimens, using the Bridgman two-dimensional compression experiment, provides accurate strains for strain components, each of which is well over 20%. The strains are accompanied by directly measured changes of volume as high as 4% due to plasticity. In such tests, *all* dimensions of the cube are continuously, and accurately, measured as the plastic strain increases. As a consequence, during unloading one also may determine the relatively small plastic strain, including the recovery of the volume to approximately its pre-deformation value. The present paper extends the same measurements of changes of volume to tubes twisted in torsion, obtaining, as shown in Table 4, numerical values comparable to those previously observed in other loading situations. The problem in torsion is somewhat more difficult, in that the torsion is observed on thin-walled tubes the inside diameters of which are not readily measurable under load. As is shown below, however, this post-deformation measurement of inside diameters has only a small influence on the result, since all other pertinent parameters—the length, the angle of twist, and even the outside diameters—can be measured before unloading begins, as well as after it has ended.

As a result of the fact that the internal constraint does not apply during unloading, one has the systematic small differences in circumferential strain for post-deformation measurements of inside and outside diameters in Table 2.

In Table 3 the values of α were the mean from the measurements of the inside and outside diameters. In fact, the determination of ϕ in eqn (9) is insensitive to this choice. For the mean α, in Table 3 the average angle was shown to be $\phi = -4.19^\circ$. From the similarly averaged α from outside diameters alone, the average angle differs by less than 0.005° from the angle $\phi = -4.19^\circ$. The same difference of 0.005° is found for the averaged α from inside diameters alone. For the mean values of α, from eqn (10) one has an average trace $\mathbf{V} = 3.0010$, as shown in Table 3. This becomes trace $\mathbf{V} = 2.9989$ for an average based only upon α determined from inside diameter measurements, and trace $\mathbf{V} = 3.0053$ for α based only upon outside diameter measurements. For the mean α, the average of the changes in volume tabulated in Table 4 is $\Delta U/U_0 = -0.014$. When α is the average from outside diameters alone, the average of the changes of volume for these tests is $\Delta U/U_0 = -0.011$, whereas when α is the average from inside diameters, the average of the changes of volume is $\Delta U/U_0 = -0.018$.

A revealing illustration of the negligible role of the rigid body rotation of principal axes is as follows: rewrite the matrix of eqn (8) as

$$V = \begin{pmatrix} \delta\cos\phi & -\delta\sin\phi & 0 \\ -\delta\sin\phi & (\alpha+\delta\sin^2\phi)\sec\phi & 0 \\ 0 & 0 & \alpha \end{pmatrix}. \tag{15}$$

In eqn (15) the component of strain related to shear is V_{xy} where $V_{xy} = E_{xy} = -\delta\sin\phi$. The corresponding shear strain is $s_{xy} = 2E_{xy}$ or $s_{xy} = -2\delta\sin\phi$. Letting $\sin\phi = \phi$, the comparable shear strain in the approximation is $s_{xy} = 2E_{xy} = -2\delta\phi$. Noting from Table 3 that $s_{xy} = \gamma$, one may write the approximation for the determination of ϕ as

$$\phi = -\gamma/2\delta. \tag{16}$$

For the 29 tests of Table 3, calculating ϕ by eqn (16), one obtains an average of $\phi = -4.16^\circ$, almost indistinguishable from $\phi = -4.19^\circ$ averaged for the same tests using eqn (9).

In sum, from the above one must conclude that for the twisting and extension to very large finite strain the role of the rigid body rotation of principal axes is negligible. To a close approximation $\mathbf{R} = \mathbf{I}$.

5. FROM KINEMATICS TO AN INTERNALLY CONSISTENT CONTINUUM THEORY

As with the analysis of the kinematics of gross deformation of beams described in a separate manuscript, the motivation for the above kinematical study was to resolve an apparent dichotomy. In 1971, on the basis of the then newly discovered internal constraint, trace $\mathbf{E} = 0$, found in experiment, and a physical theory of finite strain plasticity compatible with that constraint, both being products of many years of experiment in the present author's laboratory, Ericksen suggested an interpretation of this investigator's physical theory that led to an internally consistent continuum theory. The development of this continuum theory is described in Bell (1983a, b, 1985a, b, 1988a) and may be very briefly outlined as follows. Within the precepts of the earlier experimentally based physical theory, for *isotropic* solids in which the work done per unit volume in the undeformed reference configuration, W, depends only on the invariants of the strain tensor $\mathbf{E} = \mathbf{V} - \mathbf{I}$, a combination of the general work statement $\dot{W} = \text{trace}\, \mathbf{T}_R^T \dot{\mathbf{F}}$ and the statement for polar decomposition, $\mathbf{F} = \mathbf{VR}$, leads directly to a *symmetric* stress tensor $\boldsymbol{\sigma} = \mathbf{RT}_R^T$, where $\mathbf{T}_R$ is the first Piola–Kirchhoff stress tensor, $\mathbf{R}^{-1} = \mathbf{R}^T$ the rotation tensor, and $\mathbf{V}$ a pure homogeneous deformation.

The apparent dichotomy that generated the present series of kinematical studies arose because in experiments on the gross twisting of thin-walled tubes and on the gross bending of beams—where, for both situations, one would heretofore have anticipated a large contribution from rigid body rotation of axes—none was observed. Instead, in $\boldsymbol{\sigma} = \mathbf{RT}_R^T$, $\mathbf{R} = \mathbf{I}$ was found in experiment for all loading paths, including those of arbitrary composition and direction. The applicable stress tensor closely approximates a symmetric form of the generally non-symmetric first Piola–Kirchhoff stress tensor.

These kinematical studies have resolved the apparent dichotomy. The rotation tensor, $\mathbf{R}^{-1} = \mathbf{R}^T$, and the continuum analysis its presence implies, is present and measurable both for twisted tubes and for grossly deformed beams. The fact that $\mathbf{R}$, when present, is negligible for all stress paths, including those at very large deformation, unifies in a close approximation the internally consistent continuum theory with the experimentally based physical theory that preceded it.

Let us summarize the relevant detail from a series of recent papers (Bell, 1983a, b, 1985a, b, 1988a, b). From experiment one has the universal function $T(\Gamma)$ relating the second invariants of a total stress tensor $T^2 = 2II_S$ and the geometric strain tensor $(\mathrm{d}\Gamma)^2 = 2II_{\mathrm{d}E}$, eqn (17), the incremental constitutive statements, eqn (18), and, also from experiment, the internal constraint, eqn (19)

$$\mathrm{d}T^2/\mathrm{d}\Gamma = \beta^2 = \text{constant} \tag{17}$$

$$\mathrm{d}\mathbf{E} = 2\mathbf{S}\,\mathrm{d}T/\beta^2 \tag{18}$$

$$\text{trace}\,\mathbf{E} = 0 \tag{19}$$

$$\partial\boldsymbol{\sigma}/\partial\mathbf{X} + \rho_R\mathbf{b} = \rho_R\,\partial\mathbf{v}/\partial t \tag{20}$$

where β is a measured material constant for the ordered solid under study, and $\mathbf{S}$, the total stress, is the sum of the applied stress $\boldsymbol{\sigma} = \mathbf{RT}_R^T$ and a stress that does no work, provided by the internal constraint of eqn (19) (Bell, 1983a, 1985a), and ρ_R is the mass density in the undeformed configuration.

For specified proportional loading paths where the ratio of stress components remains constant during loading, eqn (18) can be integrated, providing

$$E_{ij} = S_{ij}T/\beta^2 + E_{ijh} \tag{21}$$

where E_{ijh} are determined constants, the intercepts on the strain axes.

For any combination of uniaxial tension and torsion along stress paths of arbitrary composition and direction, when the rigid body rotation **R** in the stress tensor $\boldsymbol{\sigma} = \mathbf{RT}_R^T$ is approximated as $\mathbf{R} = \mathbf{I}$, the incremental constitutive statements of eqn (18) reduce to

$$dE_{xx} = 4\sigma_{xx}\, dT/3\beta^2 \tag{22}$$

and

$$ds_{xy} = 4\sigma_{xy}\, dT/\beta^2. \tag{23}$$

The form of the second invariant of S in eqn (4) becomes

$$T = [(2/3)\sigma_{xx}^2 + 2\sigma_{xy}^2]^{1/2} \tag{24}$$

while the form for the second invariant of dE becomes

$$d\Gamma = [(3/2)\, dE_{xx}^2 + (1/2)\, ds_{xy}^2]^{1/2}. \tag{25}$$

If the stress path is such that one has proportional loading, i.e. the ratio of $\dot{\sigma}_{xx}/\dot{\sigma}_{xy}$ is constant, eqns (22) and (23) become

$$E_{xx} = 2\sigma_{xx} T/3\beta^2 \tag{26}$$

and

$$s_{xy} = 2\sigma_{xy} T/\beta^2 \tag{27}$$

where

$$T = [(2/3)\sigma_{xx}^2 + 2\sigma_{xy}^2]^{1/2} \quad \text{and} \quad \Gamma = [(3/2)E_{xx}^2 + (1/2)s_{xy}^2]^{1/2}. \tag{28}$$

The detailed experimental evidence for the correlation between measurement and eqns (22)–(28), assuming $\mathbf{R} = \mathbf{I}$, is as follows. Plots of E_{xx} vs s_{xy} of all of the mild steel tests of Tables 1 and 2 are found in Fig. 5 of Bell (1983b). In Figs 6 and 7 of the same paper are found $T(\Gamma)$ and T^2 vs Γ plots for the same tests. For the annealed copper tests of Tables 1 and 2, the repetition of the Taylor and Quinney experiment, test 2319, is shown in Fig. 3 of Bell (1983a). Plots of T^2 vs Γ and E_{xx} vs s_{xy} for the cyclical loading tests 2316 and 2317 are given in Figs 9–12 of that same paper. Plots of T^2 vs Γ of all of the remaining annealed copper tests of Tables 1 and 2 are given in Figs 1 and 7 of Bell and Khan (1980); E_{xx} vs s_{xy} plots for the same tests are given in Figs 2 and 8 of that paper. Of special interest is the non-proportional loading test 2211 which was illustrated in great detail in Figs 3–5 of Bell and Khan (1980).

For the maximum deformation of the tests listed in Table 3, given ϕ from eqn (9), one may calculate T and Γ in the *deformed* reference configuration. At maximum deformation in the *deformed* reference configuration the average for T and the average for Γ differ by the order of 1% from T and Γ determined in the *undeformed* reference configuration. For strains below the maxima the difference becomes unmeasurable! From eqn (15) one may calculate s_{xy} and E_{xx} in the *deformed* reference configuration to compare with measurement in the *undeformed* reference configuration. For all tests listed in Table 3, for s_{xy} and E_{xx} at maximum deformation in the deformed and undeformed reference configurations, the average differences are 0.51 and 0.65%, respectively. For the tests in combined tension–torsion alone (tests 1812–2262) the average difference for s_{xy} and E_{xx} in the deformed and undeformed reference configurations at the maximum deformation obtainable before failure are 1.46% for s_{xy} and 2.76% for E_{xx}. For strains below these maxima the difference between values in the deformed and undeformed reference configurations become unmeasurable!

From Table 1, there is no change in inside or outside diameter for tests in torsion alone, even for the largest angle of twist obtainable before buckling. Hence, at all finite strains for simple twisting the definitions of σ_{xy} and s_{xy} are given exactly by the undeformed mean radius R_m. Diametral changes occur when torsion is accompanied by axial extension, but when one introduces the deformed mean radius r_m in place of the undeformed mean radius R_m, the modifications of σ_{xy} and s_{xy} are less than 2% at maximum deformation and unmeasurable for smaller strains.

Thirty years of laboratory data that include measurements at large finite plastic strain, permit the following statements. From axial tension experiments on thin-walled tubes and on solid bars having circular and square cross-sections; from the simple twisting of thin-walled tubes; from axial tension combined with torsion in thin-walled tubes for ratios of principal stress from -1 to 0; from axial tension combined with internal pressure in thin-walled tubes for ratios of principal stress from 0 to 1; from simple compression in cubical blocks and cylindrical bars; from two-dimensional compression, the Bridgman "pure shear" experiment, introduced in 1946; from the variation of Bridgman's two-dimensional compression test introduced as such in the 1960s and now sometimes referred to as the "Channel Die" experiment; from compression and tension plastic wave experiments at high strain rates; from repetitions of and extensions of the Taylor and Quinney experiment introduced in the 1930s; and from recent experiments involving the gross deflection of cantilever and simply supported beams—the same set of incremental constitutive equations, eqns (18), with $\mathbf{R} = \mathbf{I}$, and the universal function, eqn (17), have been found to apply, whether or not rigid body rotations of principal axes are present.

For non-proportional loading, these data demonstrate that large finite plastic deformation in ordered solids is isotropic, has an internal constraint other than incompressibility, and is given by a path-dependent incremental continuum theory approximated in the undeformed reference configuration. Given the stress path, one must integrate the incremental equations to ascertain the strain components. For proportional loading, such an integration provides constitutive statements directly relating stress and strain components.

During loading along proportional stress paths, with E_{IJ} and σ_{IJ} determined as above in the *undeformed* reference configuration, and with linear response functions replaced by parabolic forms, an analogy with standard procedures in classical linear elastic theory has been found. Hence, for large finite strain one has simplified solutions to specific problems in plane stress, plane strain, and the torsion of bars of arbitrary cross-section.

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APPENDIX

The measurements of outside diameter and inside diameter that are the basis of measured E_r(max) in columns three and four of Table 2, are given in Table A1. Each outside diameter listed, whether before deformation or after deformation, is the average of ten measurements in orthogonal directions at five locations along the tube. Before annealing, the initial inside diameter was precision reamed to 0.3750 in. Each of the inside diameters is the average of eight measurements, four at 45° apart at each end. They were made by point to point telescope gages that were inserted into the open ends of the tube, allowing inside diameters to be determined in different parts of the central section.

Table A1

Test	Initial o.d. (in.)	Deformed o.d. (in.)	Initial i.d. (in.)	Deformed i.d. (in.)
1812	0.4200	0.4102	0.3742	0.3663
2211	0.4398	0.4300	0.3750	0.3648
1815	0.4195	0.4102	0.3745	0.3641
2269	0.4397	0.4290	0.3750	0.3646
1813	0.4200	0.4075	0.3745	0.3624
1805	0.4192	0.4053	0.3743	0.3605
1799	0.4192	0.4024	0.3760	0.3589
2270	0.4401	0.4195	0.3750	0.3555
1806	0.4190	0.3991	0.3740	0.3555
2316	0.4396	0.4137	0.3750	0.3525
2332	0.4150	0.3949	0.3750	0.3545
2317	0.4399	0.4135	0.3750	0.3523
2319	0.4396	0.4111	0.3750	0.3487
2286	0.4152	0.3870	0.3750	0.3494
2167	0.4151	0.3782	0.3755	0.3407
2169	0.4150	0.3720	0.3755	0.3368
2210	0.4399	0.3950	0.3750	0.3351
2271	0.4400	0.3875	0.3750	0.3292
2262	0.4396	0.3882	0.3750	0.3278
1974†	0.4391	0.3943	0.3750	0.3386

† Pure tension.